Model Theory - Lecture 9 - Prime models and categoricity

for this course only: $\aleph_0$-categoricity

Question understand "rigid" theories? for example Vector spaces over $\mathbb{F}_p$

Technology Prime models

In this lecture a theory is always on a countable language

Definition. A model of a theory is "prime" if it embeds elementarily in every other model

We will see that

1) theorem A model is prime iff it is atomic and countable

definition soon

2) If φ is such that $\forall n\ t^n_p(\varphi)$ is countable, then φ has a prime model

space of types

In this lecture we'll see - omitting types theorem

- atomic models and their relation to primeness (A)
- characterization of having a prime model (B)

(Proof)

Shifted to the next lecture

Lemma Let Φ be a theory. Assume that

- Φ is coherent and complete
- if $\Phi \models \exists x\, \varphi(x)$, then $\Phi \models \varphi(c)$ for some constant c in the language

Then, there exists a model $\mathcal{M}$ of Φ such that every element is the interpretation of some constant

Proof Φ is coherent, let $\mathcal{M}$ be a model of Φ. Let $C \subset |\mathcal{M}|$ contain all the interpretation of the constants in the language

Consider a term t in the theory. Then

$$\Phi \models \exists x\, (x = t),$$

and by the assumption on Φ, t is a constant c_t. Thus, $\mathcal{C}$ is a substructure of $\mathcal{M}$ (it contains all of the terms)

We show $\mathcal{C} \in \mathrm{Mod}(\Phi)$. Indeed, we will prove $\mathcal{C} \hookrightarrow \mathcal{M}$ is elementary, via Tarski-Vaught principle. Let

↪ "It is enough to check primitive formulas"

$$\mathcal{M} \models \exists x\, \varphi(x, a_1, \ldots, a_n), \quad \text{with } a_i \in \mathcal{C}, \text{ for } i = 1, \ldots, n,$$

by the assumption on Φ, we get

$$\mathcal{M} \models \varphi(c, a_1, \ldots, a_n), \quad \text{with } c \in \mathcal{C}$$

as $\mathcal{C}$ is the set of constants. Then $\mathcal{C} \hookrightarrow \mathcal{M}$ is elementary and we're done

∎

Some definitions before the omitting type theorem

<u>Definition</u> Let $\Sigma_1(x)$ be a 1-type. We say that $\Sigma(x)$ is "finitely supported" if there is a formula θ that implies all the formulas in $\Sigma(x)$

<u>Remark</u>
- Note that finitely supported is not the same as complete
- Because we work in FOL we just need one θ instead of a finite amount (in other logic we might not have finite conjunction).

<u>Definition</u> A type is "omitted" if there exists a model that does not model it

We move to the OTT.

Theorem (Omitting Type) T coherent, $\Sigma(x_1, \dots, x_n)$ an n-type that is not finitely supported There exists a model which omits Σ

Proof Let $\{c_1, \dots\}$ be an enumeration of the constants in $\mathcal{L}$ and $\{\varphi_1, \dots\}$ an enumeration of the formulas of $\mathcal{L}$ Let $\{k_1, \dots\}$ be a countable set of fresh constants We want to build

$$T_0 = T, \quad T_0 \subset T_1 \subset \dots \subset \bigcup_{n \in \mathbb{N}} T_n = \overline{T}, \quad \text{all in language } \mathcal{L} \cup \{k_n\}_{n \in \mathbb{N}}$$

so that $\overline{T}$ satisfies
- completeness and coherency
- $\overline{T} \models \exists x\, \varphi(x) \rightsquigarrow \overline{T} \models \varphi(c)$ for some constant c,
- for all constants c, there exists $\delta(x) \in \Sigma(x)$ such that $\overline{T} \models \neg \delta(c)$

To do so, let $T_0 = T$ Now assume T_n is defined If $T_n \cup \{\varphi_n\}$ is coherent, define $T_n' = T_n \cup \{\varphi_n\}$, otherwise $T_n' = T_n \cup \{\neg \varphi_n\}$

Moreover, if $\varphi_n \equiv \exists x\, \psi$ for some formula ψ, then

up to reordering

$$T_n'' = T_n' \cup \{\exists x\, \psi \to \psi(k_n)\}$$

and let $T_n'' = T_n'$ if φ_n is not of said form We claim there exists a formula $\chi(x_1, \dots, x_n)$ such that $T_n'' = T_n \cup \{\chi(x_1, \dots, x_n)\}$ just take the $\wedge$

Since $\Sigma(x)$ is not finitely supported, T_n'' cannot prove all the formulas in $\Sigma(x)$. Then, there exists $\delta(x) \in \Sigma(x)$ such that $T_{n+1} = T_n'' \cup \{\neg \delta(x)\}$ is coherent It is easy to prove $\overline{T}$ satisfies the properties for the lemma and a model for $\overline{T}$ omits $\Sigma(x)$ ∎

<u>Definition</u> A complete type Σ is "principal" if there is a formula ψ such that $V_\psi = \{\Sigma\}$ ↖ i.e. it is an isolated point in the space of types

see the opens ↲
of the topology on the space of types

<u>Definition</u> A structure $\mathfrak{M}$ in a theory φ is "atomic" if every finite tuple $(a_i) \in |\mathfrak{M}|^n$ realizes a principal type for φ

<u>Theorem</u> Countable atomic models are all isomorphic, when φ complete.

<u>Proof</u> This goes by B&F starting from the empty partial isomorphism. Then, consider $(\mathfrak{M}, a_1, \dots, a_n) \equiv_p (\mathcal{N}, b_1, \dots, b_n)$ Let $c \in |\mathfrak{M}|$ new We prove that, if the type of c over $a_1, \dots, a_n$ is isolated, then we can extend the isomorphism This is just atomicity. Let $\Sigma(x)$ be that type Moreover, it is realized, since φ is complete This completes the proof, since we choose the realizer in $\mathcal{N}$ to link to c

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<u>Scholium</u> Countable atomic models are prime

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↓
Corollary of the proof